ON FALTINGS' ANNIHILATOR THEOREM

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Dedicated to Professor Shiro Goto on the occasion of his sixtieth birthday

Abstract. In the present article, the author shows that Faltings' annihilator theorem holds for any Noetherian ring A if A is universally catenary; all the formal fibers of all the localizations of A are Cohen-Macaulay; and the Cohen-Macaulay locus of each finitely generated A-algebra is open.

1. Introduction

Throughout the present article, A always denotes a commutative Noetherian ring. We say that the annihilator theorem holds for A if it satisfies the following proposition [4].

The Annihilator Theorem. *Let M be a finitely generated A-module, n an integer and Y, Z subsets of* $\operatorname{Spec} A$ *which are stable under specialization. Then the following statements are equivalent*:

(1) $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_{\mathfrak{q}} \geq n$ *for any* $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ *and* $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$;
(2) *there is an ideal $\mathfrak{b}$ in A such that $V(\mathfrak{b}) \subset Y$ and $\mathfrak{b}$ annihilates local cohomology modules $H^0_Z(M)$, ..., $H^{n-1}_Z(M)$.*

Faltings [3] proved that the annihilator theorem holds for A if A has a dualizing complex or if A is a homomorphic image of a regular ring and that (2) always implies (1). Several authors [1, 2, 9, 10, 11] tried to improve Faltings' result. In this article, the author shows the following

Theorem 1.1. *The annihilator theorem holds for A if*

(C1) *A is universally catenary*;
(C2) *all the formal fibers of all the localizations of A are Cohen-Macaulay*; *and*
(C3) *the Cohen-Macaulay locus of each finitely generated A-algebra is open.*

These conditions are not only sufficient but also necessary for the annihilator theorem. Indeed, Faltings [4] showed that A satisfies (C1)–(C3) whenever the annihilator theorem holds for each essentially of finite type A-algebra.

These conditions are also related to the uniform Artin-Rees theorem and the uniform Briançon-Skoda theorem. We give an affirmative answer to the conjecture of Huneke [7, Conjecture 2.13] in the last section.

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2. Preliminaries

First we recall the definition of the local cohomology functor. A subset Z of $\operatorname{Spec} A$ is said to be stable under specialization if $\mathfrak{p} \in Z$ implies $V(\mathfrak{p}) \subset Z$. Let M be an A-module and Z a subset of $\operatorname{Spec} A$ which is stable under specialization. Then we put

$$H^0_Z(M) = \{m \in M \mid \operatorname{Supp} Am \subset Z\}.$$

It is an A-submodule of M and $H^0_Z(-)$ is a left exact functor.

Definition 2.1 ([5, p. 223])**.** The local cohomology functor $H^p_Z(-)$ with respect to Z is the right derived functor of $H^0_Z(-)$.

If $\mathfrak{b}$ is an ideal, then $Z = V(\mathfrak{b})$ is stable under specialization and $H^p_Z(-)$ coincides with the ordinary local cohomology functor $H^p_{\mathfrak{b}}(-)$.

Let Z be a subset of $\operatorname{Spec} A$ which is stable under specialization. If $\mathfrak{b}$, $\mathfrak{b}'$ are ideals such that $V(\mathfrak{b})$, $V(\mathfrak{b}') \subset Z$, then $V(\mathfrak{b} \cap \mathfrak{b}') \subset Z$. Therefore the set $\mathcal{F}$ of all ideals $\mathfrak{b}$ such that $V(\mathfrak{b}) \subset Z$ is a directed set with respect to the opposite inclusion. If $\mathfrak{b}$, $\mathfrak{b}' \in \mathcal{F}$ such that $\mathfrak{b}' \subset \mathfrak{b}$, then there is a natural transformation $\operatorname{Ext}^p_A(A/\mathfrak{b}, -) \to \operatorname{Ext}^p_A(A/\mathfrak{b}', -)$. Since $H^0_Z(-) = \operatorname{inj\,lim}_{\mathfrak{b} \in \mathcal{F}} \operatorname{Hom}(A/\mathfrak{b}, -)$, we obtain the natural isomorphism

$$H^p_Z(-) = \operatorname*{inj\,lim}_{\mathfrak{b} \in \mathcal{F}} \operatorname{Ext}^p_A(A/\mathfrak{b}, -). \tag{2.1.1}$$

The following lemma was essentially given by Raghavan [11, p. 491].

Lemma 2.2. *Let M be a finitely generated A-module. Then $\mathcal{L} = \{H^0_Z(M) \mid Z \subset \operatorname{Spec} A$ is stable under specialization$\}$ is a finite set.*

Proof. Let $\operatorname{Ass} M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ and $0 = M_1 \cap \dots \cap M_r$ be a primary decomposition of 0 in M where $\operatorname{Ass} M/M_i = \{\mathfrak{p}_i\}$ for all i. Then $H^0_Z(M) = \bigcup_{V(\mathfrak{b}) \subset Z} 0 :_M \mathfrak{b} = \bigcap_{\mathfrak{p}_i \notin Z} M_i$. Therefore $\# \mathcal{L} \leq 2^r$. □

We need Cousin complexes to prove Theorem 1.1.

Let M be a finitely generated A-module. For a prime ideal $\mathfrak{p} \in \operatorname{Supp} M$, the M-height of $\mathfrak{p}$ is defined to be $\operatorname{ht}_M \mathfrak{p} = \dim M_{\mathfrak{p}}$. If $\mathfrak{b}$ is an ideal in A such that $M \neq \mathfrak{b} M$, then let $\operatorname{ht}_M \mathfrak{b} = \inf\{\operatorname{ht}_M \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M \cap V(\mathfrak{b})\}$.

Definition 2.3 ([12])**.** The Cousin complex $(M^\bullet, d^\bullet_M)$ of M is defined as follows:

Let $M^{-2} = 0$, $M^{-1} = M$ and $d^{-2}_M \colon M^{-2} \to M^{-1}$ be the zero map. If $p \geq 0$ and $d^{p-2}_M \colon M^{p-2} \to M^{p-1}$ is given, then we put

$$M^p = \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Supp} M \\ \operatorname{ht}_M \mathfrak{p} = p}} (\operatorname{Coker} d^{p-2}_M)_{\mathfrak{p}}.$$

If $\xi \in M^{p-1}$ and $\bar{\xi}$ is the image of ξ in $\operatorname{Coker} d^{p-2}_M$, then the component of $d^p_M(\xi)$ in $(\operatorname{Coker} d^{p-2}_M)_{\mathfrak{p}}$ is $\bar{\xi}/1$.

The following theorem contains [6, Theorems 11.4 and 11.5].

Theorem 2.4. *Assume that A satisfies* (C1)–(C3) *and let M be a finitely generated A-module satisfying*

(QU) $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{ht}_M \mathfrak{q} = \operatorname{ht}_M \mathfrak{p}$ *for any* $\mathfrak{p}$, $\mathfrak{q} \in \operatorname{Supp} M$ *such that* $\mathfrak{p} \supset \mathfrak{q}$.

Then there is an ideal $\mathfrak{a}$ *in* A *satisfying the following properties*:

(1) $V(\mathfrak{a})$ *is the non-Cohen-Macaulay locus of* M. *In particular,* $\operatorname{ht}_M \mathfrak{a} > 0$.

(2) *Let* Z *be a subset of* $\operatorname{Spec} A$ *which are stable under specialization and* n *an integer. If* $\operatorname{ht}_M \mathfrak{p} \geq n$ *for any* $\mathfrak{p} \in Z \cap \operatorname{Supp} M$, *then* $\mathfrak{a} H^p_Z(M) = 0$ *for each* $p < n$.

(3) *Let* $x_1, \dots, x_n \in A$ *be a sequence. If* $\operatorname{ht}_M(x_1, \dots, x_n)A \geq n$, *then* $\mathfrak{a}$ *annihilates the Koszul cohomology module* $H^p(x_1, \dots, x_n; M)$ *of* M *with respect to* $x_1, \dots, x_n$ *for any* $p < n$.

Proof. Let $M^\bullet$ be the Cousin complex of M and $\mathfrak{a}$ the product of all the annihilators of all the non-zero cohomologies of $M^\bullet$. Then it is well-defined and satisfies (1). See [8, Corollary 6.4].

We prove (2). Because of (2.1.1), it is enough to show that $\mathfrak{a} \operatorname{Ext}^p(A/\mathfrak{b}, M) = 0$ for any ideal $\mathfrak{b}$ such that $V(\mathfrak{b}) \subset Z$ and for any $p < n$. Let $\mathfrak{b}$ be such an ideal and $F_\bullet$ a free resolution of $A/\mathfrak{b}$. The double complex $\operatorname{Hom}(F_\bullet, M^\bullet)$ gives two spectral sequences

$$'E_2^{pq} = \operatorname{Ext}^p(A/\mathfrak{b}, H^q(M^\bullet)) \Rightarrow H^{p+q}(\operatorname{Hom}(F_\bullet, M^\bullet)),$$
$$''E_2^{pq} = H^p(\operatorname{Ext}^q(A/\mathfrak{b}, M^\bullet)) \Rightarrow H^{p+q}(\operatorname{Hom}(F_\bullet, M^\bullet)).$$

The first spectral sequence tells us that $\mathfrak{a} H^k(\operatorname{Hom}(F_\bullet, M^\bullet)) = 0$ for any k.

On the other hand, $''E_2^{pq} = 0$ if $p < -1$ or if $q < 0$. Let $0 \leq p < n$ be an integer and $\mathfrak{p} \in \operatorname{Supp} M$ such that $\operatorname{ht}_M \mathfrak{p} = p$. Since $\mathfrak{b} \not\subset \mathfrak{p}$, we find that $\operatorname{Hom}(F_\bullet, (\operatorname{Coker} d_M^{p-2})_\mathfrak{p})$ is exact. Hence $\operatorname{Hom}(F_\bullet, M^p)$ is also exact. Thus $''E_2^{pq} = 0$ if $0 \leq p < n$ and $''E_2^{-1,q} = \operatorname{Ext}^q(A/\mathfrak{b}, M)$. If $k < n$, then $''E_2^{p,k-p-1} = {}''E_2^{p,k-p} = 0$ whenever $p \neq -1$. Therefore $H^{k-1}(\operatorname{Hom}(F_\bullet, M^\bullet)) = {}''E_2^{-1,k} = \operatorname{Ext}^k(A/\mathfrak{b}, M)$ is annihilated by $\mathfrak{a}$.

Next we consider (3). Let $K_\bullet$ be the Koszul complex of A with respect to $x_1, \dots, x_n$. By considering the double complex $\operatorname{Hom}(K_\bullet, M^\bullet)$, instead of $\operatorname{Hom}(F_\bullet, M^\bullet)$, we obtain the assertion. □

3. The proof of Theorem 1.1

Before the proof of Theorem 1.1, we fix some notation. Let $\mathfrak{X}$ be the free Abelian group with basis $\operatorname{Spec} A$ and $\mathfrak{X}_+ = \{\sum k_\mathfrak{p} \mathfrak{p} \mid k_\mathfrak{p} \geq 0 \text{ for all } \mathfrak{p}\}$. If $\alpha = k_1\mathfrak{p}_1 + \cdots + k_n\mathfrak{p}_n$ and $\beta = l_1\mathfrak{p}_1 + \cdots + l_n\mathfrak{p}_n$ where $\mathfrak{p}_i \neq \mathfrak{p}_j$ whenever $i \neq j$, then we put

$$\alpha \vee \beta = \sum_{i=1}^n \max\{k_i, l_i\}\mathfrak{p}_i.$$

It is clear that $(\alpha \vee \beta) + \gamma = (\alpha + \gamma) \vee (\beta + \gamma)$. Let $\alpha = k_1\mathfrak{p}_1 + \cdots + k_n\mathfrak{p}_n \in \mathfrak{X}_+$ and Y be a subset of $\operatorname{Spec} A$ which is stable under specialization. Then we put $\mathfrak{b}(\alpha, Y) = \prod_{\mathfrak{p}_i \in Y} \mathfrak{p}_i^{k_i}$. Since $V(\mathfrak{b}(\alpha, Y)) \subset Y$, Theorem 1.1 is contained in the following

Theorem 3.1. *Assume that* A *satisfies* (C1)–(C3). *If* M *is a finitely generated* A*-module, then there is* $\alpha(M) \in \mathfrak{X}_+$ *satisfying the following property*:

Let Y, Z *be subsets of* $\operatorname{Spec} A$ *which are stable under specialization and* n *an integer. If*

(A) $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_\mathfrak{q} \geq n$ *for any* $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ *and* $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$,

then

(B) $\mathfrak{b}(\alpha(M), Y)$ *annihilates* $H^0_Z(M), \dots, H^{n-1}_Z(M)$.

We prove this theorem by the Noetherian induction on $\operatorname{Supp} M$ and the induction on the number of associated primes of M.

If $M = 0$, then $\alpha(M) = 0$ obviously satisfies the assertion. Assume that $M \neq 0$ and that, for any finitely generated A-module M', there is $\alpha(M')$ satisfying the assertion of Theorem 3.1 if $\operatorname{Supp} M' \subsetneq \operatorname{Supp} M$ or if $\operatorname{Supp} M' = \operatorname{Supp} M$ and $\# \operatorname{Ass} M' < \# \operatorname{Ass} M$. We first prove the following claim.

Claim. *There is $\alpha'(M) \in \mathfrak{X}_+$ satisfying the following property*:

Let Y, Z be subsets of $\operatorname{Spec} A$ *which are stable under specialization and n an integer. If $Y \cap \operatorname{Ass} M = \emptyset$ and* (A) *holds, then* (B) *also does.*

Proof. Let $\operatorname{Ass} M = \{P_1, \dots, P_r\}$. We may assume that $P_1 \not\subset P_2, \dots, P_r$ without loss of generality. There is an exact sequence

$$0 \to L \to M \to N \to 0$$

such that $\operatorname{Ass} L = \{P_2, \dots, P_r\}$ and $\operatorname{Ass} N = \{P_1\}$. Since A is universally catenary and N has the unique minimal prime, N satisfies (QU). Let $\mathfrak{a}$ be the ideal obtained by applying Theorem 2.4 to N. Then $P_1 \subsetneq \mathfrak{a}$. Since $P_1 \not\subset P_2, \dots, P_r$, we find that $\mathfrak{a} \not\subset P_2, \dots, P_r$. Let $x'' \in \mathfrak{a} \setminus (P_1 \cup \cdots \cup P_r)$.

Since $\operatorname{Supp} L \subsetneq \operatorname{Supp} M$ or since $\operatorname{Supp} L = \operatorname{Supp} M$ and $\# \operatorname{Ass} L < \# \operatorname{Ass} M$, there is $\alpha(L) \in \mathfrak{X}_+$ satisfying the assertion of Theorem 3.1. Let $\alpha(L) = k_1 Q_1 + \cdots + k_s Q_s$. We may assume that $Q_1, \dots, Q_{s_0} \not\subset P_1 \cup \cdots \cup P_r$ and $Q_{s_0+1}, \dots, Q_s \subset P_1 \cup \cdots \cup P_r$. Let $x' \in Q_1^{k_1} \cdots Q_{s_0}^{k_{s_0}} \setminus P_1 \cup \cdots \cup P_r$ and $x = x'x''$.

Since x is an M-non zero divisor, $\operatorname{Supp} M/xM \subsetneq \operatorname{Supp} M$. We want to show that $\alpha'(M) = \alpha(M/xM)$ satisfies the assertion of the claim.

Let Y, Z be subsets of $\operatorname{Spec} A$ which are stable under specialization and n an integer. Assume that $Y \cap \operatorname{Ass} M = \emptyset$ and $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_{\mathfrak{q}} \geq n$ for any $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ and $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$. If $\mathfrak{p} \in Z \cap \operatorname{Supp} N$, then $\operatorname{ht} \mathfrak{p}/P_1 + \operatorname{depth} M_{P_1} \geq n$ because $\operatorname{Supp} N = V(P_1)$ and $P_1 \notin Y$. Since $\operatorname{depth} M_{P_1} = 0$, we have

$$\operatorname{ht}_N \mathfrak{p} = \operatorname{ht} \mathfrak{p}/P_1 \geq n \quad \text{for any } \mathfrak{p} \in Z \cap \operatorname{Supp} N. \tag{3.1.1}$$

By using Theorem 2.4 (2), we find that $x'' H_Z^p(N) = 0$ for any $p < n$.

Let $\mathfrak{q} \in \operatorname{Spec} A \setminus (Y \cup V(x''A))$ and $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$. Since $x'' \notin \mathfrak{q}$, $N_{\mathfrak{q}}$ is Cohen-Macaulay. If $N_{\mathfrak{q}} \neq 0$, then $\mathfrak{p} \in Z \cap \operatorname{Supp} N$ and hence

$$\begin{aligned} \operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} N_{\mathfrak{q}} &= \operatorname{ht} \mathfrak{p}/\mathfrak{q} + \dim N_{\mathfrak{q}} \\ &= \operatorname{ht}_N \mathfrak{p} \geq n. \end{aligned}$$

Here we used (3.1.1). If $N_{\mathfrak{q}} = 0$, then $\operatorname{depth} N_{\mathfrak{q}} = \infty$ and hence $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} N_{\mathfrak{q}} \geq n$. Since $\mathfrak{q} \notin Y$, the assumption tells us that $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_{\mathfrak{q}} \geq n$. Therefore $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} L_{\mathfrak{q}} \geq n$. Because of the induction hypothesis,

$$\mathfrak{b}(\alpha(L), Y \cup V(x''A)) H_Z^p(L) = 0$$

for $p < n$.

Since $x'' \notin P_1 \cup \cdots \cup P_r$, $P_1, \dots, P_r \notin Y$ and $Q_{s_0+1}, \dots, Q_s \subset P_1 \cup \cdots \cup P_r$, we have $Q_{s_0+1}, \dots, Q_s \notin Y \cup V(x''A)$. Therefore $x' \in Q_1^{k_1} \cdots Q_{s_0}^{k_{s_0}} \subset \mathfrak{b}(\alpha(L), Y \cup V(x''A))$ and hence $x' H_Z^p(L) = 0$ if $p < n$. Since $H_Z^p(L) \to H_Z^p(M) \to H_Z^p(N)$ is exact, $x H_Z^p(M) = 0$ if $p < n$.

Since x is an M-non zero divisor, $H_Z^0(M) = 0$,

$$0 \to H_Z^{p-1}(M) \to H_Z^{p-1}(M/xM) \to H_Z^p(M) \to 0$$

is exact for $p < n$ and $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth}(M/xM)_\mathfrak{q} \geq n-1$ for any $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ and $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$. Therefore $\mathfrak{b}(\alpha'(M), Y) = \mathfrak{b}(\alpha(M/xM), Y)$ annihilates $H_Z^p(M)$ if $p < n$. □

Next we give $\alpha(M)$. Let $\operatorname{Ass} M = \{P_1, \dots, P_r\}$ and $0 = M_1 \cap \cdots \cap M_r$ be a primary decomposition of 0 in M such that $\operatorname{Ass} M/M_i = \{P_i\}$. Then there are integers $k_1, \dots, k_r$ such that $P_i^{k_i} M \subset M_i$ for each i.

Let $\{H_Z^0(M) \mid Y \subset \operatorname{Spec} A \text{ is stable under specialization}\} = \{L_1, \dots, L_s\}$. Assume that $L_1 = 0$ and $L_2, \dots, L_s \neq 0$. Since $\operatorname{Supp} M/L_i \subsetneq \operatorname{Supp} M$ or $\operatorname{Supp} M/L_i = \operatorname{Supp} M$, $\# \operatorname{Ass} M/L_i < \# \operatorname{Ass} M$, there is $\alpha(M/L_i) \in \mathfrak{X}_+$ satisfying the assertion of Theorem 3.1 for each $i = 2, \dots, s$. We put $\alpha(M) = \alpha'(M) \vee [\sum k_i P_i + \alpha(M/L_2) \vee \cdots \vee \alpha(M/L_s)]$. Then $\alpha(M)$ has required property.

Indeed, let Y, Z be subsets of $\operatorname{Spec} A$ which are stable under specialization and n an integer. If $H_Y^0(M) = 0$, then $Y \cap \operatorname{Ass} M = \emptyset$ and hence $\mathfrak{b}(\alpha'(M), Y)$ annihilates $H_Z^0(M), \dots, H_Z^{n-1}(M)$. Assume that $H_Y^0(M) = L_j$ for some $2 \leq j \leq s$. If $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ and $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$, then $(L_j)_\mathfrak{q} = 0$ and hence $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth}(M/L_j)_\mathfrak{q} = \operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_\mathfrak{q} \geq n$. Therefore $\mathfrak{b}(\alpha(M/L_j), Y)$ annihilates $H_Z^0(M/L_j), \dots, H_Z^{n-1}(M/L_j)$. On the other hand, since there is a monomorphism

$$L_j = \bigcap_{P_i \notin Y} M_i \hookrightarrow \bigoplus_{P_i \in Y} M/M_i,$$

we find that $\mathfrak{b}(\sum k_i P_i, Y) L_j = 0$. Since $H_Z^p(L_j) \to H_Z^p(M) \to H_Z^p(M/L_j)$ is exact, $\mathfrak{b}(\sum k_i P_i + \alpha(M/L_j), Y)$ annihilates $H_Z^0(M), \dots, H_Z^{n-1}(M)$. Thus (B) holds.

If $L_1, \dots, L_s$ are all non-zero, then we put $\alpha(M) = \sum k_i P_i + \alpha(M/L_1) \vee \cdots \vee \alpha(M/L_s)$. We can show that $\alpha(M)$ satisfies the assertion of Theorem 3.1 in the same way as above. The proof of Theorem 1.1 is completed.

The following corollary is an improvement of [11, Theorem 3.1].

Corollary 3.2. *Assume that A satisfies* (C1)–(C3). *If M is a finitely generated A-module, then there is a positive integer k satisfying the following property*:

Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals in A and n an integer. If $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_\mathfrak{q} \geq n$ for any $\mathfrak{q} \in \operatorname{Spec} A \setminus V(\mathfrak{b})$ and $\mathfrak{p} \in V(\mathfrak{a} + \mathfrak{q})$, then $\mathfrak{b}^k H_\mathfrak{a}^p(M) = 0$ for all $p < n$.

Proof. Let $\alpha(M) = k_1 \mathfrak{p}_1 + \cdots + k_r \mathfrak{p}_r$ and $k = k_1 + \cdots + k_r$. Then $\mathfrak{b}(\alpha(M), V(\mathfrak{b})) \supset \mathfrak{b}^k$. □

4. A conjecture of Huneke

The following theorem is an affirmative answer to Conjecture 2.13 of [7]. Its proof is similar to that of Theorem 2.4.

Theorem 4.1. *Assume that A satisfies* (C1)–(C3) *and let M be a finitely generated A-module satisfying* (QU). *Then there is an ideal $\mathfrak{a}$ in A satisfying the following property*:

(1) $\operatorname{ht}_M \mathfrak{a} > 0$.

(2) *Let*

$$0 \longrightarrow F^{-n} \xrightarrow{f^{-n}} F^{-n+1} \longrightarrow \cdots \longrightarrow F^{-1} \xrightarrow{f^{-1}} F^0$$

be a complex of finitely generated free A-modules such that

(a) $\operatorname{rank} f^{-n} = \operatorname{rank} F^{-n}$;

(b) $\operatorname{rank} F^i = \operatorname{rank} f^i + \operatorname{rank} f^{i-1}$ *for each* $-n < i < 0$;

(c) $\operatorname{ht}_M I_{r_i}(f^i) \geq -i$ *for each* $-n \leq i < 0$ *where* $r_i = \operatorname{rank} f_i$ *for each* i.
Then $\mathfrak{a}H^p(F^\bullet \otimes M) = 0$ *for all* $p < 0$. *Here* $I_{r_i}(f^i)$ *denotes the ideal generated by all the* r_i*-minors of the representation matrix of* f^i.

Proof. Let $M^\bullet$ be the Cousin complex of M and $\mathfrak{a}$ the product of all the annihilators of all the non-zero cohomologies of $M^\bullet$. Then $\mathfrak{a}$ satisfies (1). The double complex $F^\bullet \otimes M^\bullet$ gives a spectral sequence

$$'E_2^{pq} = H^p(F^\bullet \otimes H^q(M^\bullet)) \Rightarrow H^{p+q}(F^\bullet \otimes M^\bullet).$$

It tells us that $\mathfrak{a}H^p(F^\bullet \otimes M^\bullet) = 0$ for all p. On the other hand, $F^\bullet \otimes M^\bullet$ gives another spectral sequence $''E_2^{pq} \Rightarrow H^{p+q}(F^\bullet \otimes M^\bullet)$ where $''E_2^{pq}$ is the cohomology of

$$H^q(F^\bullet \otimes M^{p-1}) \to H^q(F^\bullet \otimes M^p) \to H^q(F^\bullet \otimes M^{p+1}).$$

If $0 \leq p < n$ and $\mathfrak{p} \in \operatorname{Supp} M$ such that $p = \operatorname{ht}_M \mathfrak{p}$, then

$$0 \longrightarrow (F^{-n})_\mathfrak{p} \longrightarrow \cdots \longrightarrow (F^{-p})_\mathfrak{p}$$

is split exact and hence $H^q(F^\bullet \otimes M^p) = 0$ if $q < -p$. Therefore $''E_2^{pq} = 0$ if $p > 0$ and $p + q < 0$. Furthermore $''E_2^{-1,q} = H^q(F^\bullet \otimes M)$ for each $q < 0$. Of course, $''E_2^{pq} = 0$ if $p < -1$. Thus $H^p(F^\bullet \otimes M) = {''E_2^{-1,p}} = H^{p-1}(F^\bullet \otimes M^\bullet)$ is annihilated by $\mathfrak{a}$ if $p < 0$. □

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